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The Existence of Solution in $H^1(R^N)$ for Nonclassical Diffusion Equations

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Keywords : *Nonclassical diffusion equations; Weak solution; Absorbing set.*

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THE EXISTENCE OF SOLUTION IN H1RN FOR NONCLASSICAL DIFFUSION EQUATIONS

Strictly as per the compliance and regulations of :

Ref.
[1] Aifantis. E. C, *On the problem of diffusion in solids*, Acta Mech, **37** (1980):265-296.
[2] Lions. J. L, Magenes.E, *Non-homogeneous boundary value problems and applications*, Springverlag,Berlin, (1972).

The Existence of Solution in $H^1(R^N)$ for Nonclassical Diffusion Equations

LIU Yong-feng[α] & MA Qiao-zhen[σ]

Abstract - In this paper, we prove the existence of weak solution for a nonclassical diffusion equations in $H^1(R^N)$. The result in this part are new.
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I. Introduction

In this paper, we investigate the following nonclassical diffusion equations

$$u_t - \Delta u_t - \Delta u + f(x,u) = g(x), \quad x \in R^N, \tag{1.1}$$

with the initial data

$$u(x,0) = u_0, \quad x \in R^N. \tag{1.2}$$

This equation is a special form of the nonclassical diffusion equation used in fluid mechanics, solid mechanics and heat conduction theory(see [1, 2]). On bounded domains, the long-time behavior have been discussed by many authors in [3-11].

To our best knowledge, the existence of weak solution in R^N for the nonclassical diffusion equation have not been considered by predecessors.

In this paper, we consider the existence of weak solution in $H^1(R^N)$ if $g(x) \in L^2(R^N)$, and the nonlinearity $f(x,u) = f_1(u) + a(x)f_2(u)$ satisfies:

(F_1) $\alpha_1 \mid u \mid^p - \beta_1 \mid u \mid^2 \leq f_1(u)(u) \leq \gamma_1 \mid u \mid^p + \delta_1 \mid u \mid^2, f_1(u)u \geq 0, p \geq 2,$ *and* $f_1'(u) \geq -c;$

(F_2) $\alpha_2 \mid u \mid^p - \beta_2 \leq f_2(u)(u) \leq \gamma_2 \mid u \mid^p + \delta_2, p \geq 2,$ *and* $f_2'(u) \geq -c;$

and

(A) $a \in L^1(R^N) \cap L^\infty(R^N),\ a(x) > 0.$

where $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i,\ i = 1, 2$, and c are all positive constants.

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II. Unique Weak Solution

Lemma 2.1 *([11]) Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0,T;X)$, and $\mathrm{d}u_n/\mathrm{d}t$ is uniformly bounded in $L^p(0,T;Y)$, for some $p>1$. Then there is a subsequence that converges strongly in $L^2(0,T;H)$.*

Theorem 2.1 *Assume (F_1), (F_2) and (A) are satisfied. Then for any $T>0$ and $u_0 \in H^1(R^N)$, there is a unique solution u of $(1.1)-(1.2)$ such that*

$$u \in \mathcal{C}^1([0,T];H^1(R^N)) \cap L^p(0,T;L^p(R^N)).$$

Moreover, the solution continuously depends on the initial data.

Proof We divide into three steps:

Step 1 For any $n \in N$, we consider the existence of the weak solution for the following problem in $B(0,n) \triangleq B_n \subset R^N$,

$$u_t - \Delta u_t - \Delta u + f(x,u) = g(x), \quad x \in B_n, \tag{2.1}$$

$$u(x,0) = u_0 \in H^1(B_n). \tag{2.2}$$

$$u\mid_{\partial\Omega} = 0. \tag{2.3}$$

Choose a smooth function $\chi_n(x)$ satisfy

$$\chi_n(x) = \begin{cases} 1, & x \in B_{n-1}, \\ 0, & x \notin B_n. \end{cases} \tag{2.4}$$

Since B_n is a bounded domain, so the existence and uniqueness of solutions can be obtained by the standard Faedo-Galerkin methods, see [3,5,8,11], we have the unique weak solution

$$u_n \in \mathcal{C}^1([0,T];H^1(B_n)) \cap L^p(0,T;L^p(B_n)) \ \ and\ u_n(x,0) = \chi_n(x)u_0(x).$$

Step 2 According to Step 1, and we denote $\frac{\mathrm{d}}{\mathrm{d}t}u_n = u_{nt}$, then u_n satisfy

$$u_{nt} - \Delta u_{nt} - \Delta u_n + f(x,u_n) = g(x), \quad x \in B_n, \tag{2.5}$$

$$u_n(x,0) = \chi_n(x)u_0(x), \tag{2.6}$$

$$u_n\mid_{\partial B_n} = 0. \tag{2.7}$$

For the mathematical setting of the problem, we denote $\|\cdot\|_{L^2(B_n)} \triangleq \|\cdot\|_{B_n}$, $\|\cdot\|_{L^1(R^N)} \triangleq \|\cdot\|_1$, $\|\cdot\|_{L^2(R^N)} \triangleq \|\cdot\|$, $\|\cdot\|_{L^\infty(R^N)} \triangleq \|\cdot\|_\infty$.

Multiply (2.5) by u_n in B_n, using $f_1(u)u \geq 0$, (F_2) and (A), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u_n\|_{B_n}^2 + \|u_n\|_{B_n}^2) + \|\nabla u_n\|_{B_n}^2 \quad &\leq \int_{B_n} a(x)(\beta_2 - \alpha_2 \mid u\mid^p)\mathrm{d}x + \int_{B_n} g u_n \mathrm{d}x \\ &\leq \beta_2 \|a(x)\|_1 - \int_{B_n} \alpha_2 a(x)\mid u\mid^p \mathrm{d}x + \frac{\|g\|^2}{2\lambda} + \frac{\lambda}{2}\|u_n\|_{B_n}^2\end{aligned}$$

By the Poincaré inequality, for some $\nu > 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u_n\|_{B_n}^2 + \|u_n\|_{B_n}^2) + \nu(\|\nabla u_n\|_{B_n}^2 + \|u_n\|_{B_n}^2) \quad + \int_{B_n} \alpha_2 a(x)\mid u\mid^p \mathrm{d}x$$

Ref.

[3] V. K. Kalantarov, *On the attractors for some non-linear problems of mathematical physics*, Zap. Nauch. Sem. LOMI **152** (1986):50-54.
[11] C.Robinson,*Infinite-dimensional Dynamincal Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global attractors*, Cambridge Universlty Press(2001).

$$\leq \quad \beta_2 \parallel a(x) \parallel_1 + \frac{\parallel g \parallel^2}{2\lambda}. \tag{2.8}$$

Hence, we have

$$\parallel \nabla u_n(T) \parallel_{B_n}^2 + \|u_n(T)\|_{B_n}^2 + 2\nu \int_0^T (\parallel \nabla u_n(T) \parallel_{B_n}^2 \quad + \quad \|u_n(T)\|_{B_n}^2) + 2\int_0^T \int_{B_n} \alpha_2 a(x) \mid u \mid^p \mathrm{d}x$$
$$\leq \quad (2\beta_2 \parallel a(x) \parallel_1 + \frac{\parallel g \parallel^2}{\lambda})T. \tag{2.9}$$

Notes

We get the following estimate:

$$\sup_{t\in[0,T]} \parallel \nabla u_n(t) \parallel_{B_n}^2 + \|u_n(t)\|_{B_n}^2 \leq C, \tag{2.10}$$

$$\int_0^T (\parallel \nabla u_n(t) \parallel_{B_n}^2 + \|u_n(t)\|_{B_n}^2) \leq C, \tag{2.11}$$

$$\int_0^T \int_{B_n} \alpha_2 a(x) \mid u(t) \mid^p \mathrm{d}x \leq C, \tag{2.12}$$

Similar to (2.8), using (F_1), (F_2) and (A), we have

$$\int_0^T \int_{B_n} \mid u(t) \mid^p \mathrm{d}x \leq C. \tag{2.13}$$

where C is independent on n.

According to (F_1) and (F_2), we have

$$\mid f_1(u_n) \mid \leq C(\mid u_n \mid^{p-1} + \mid u_n \mid). \tag{2.14}$$

$$\mid f_2(u_n) \mid \leq C(\mid u_n \mid^{p-1} + 1). \tag{2.15}$$

We choose q such that $\frac{1}{p} + \frac{1}{q} = 1$, then $(p-1)q = p$. Noting that $p \geq 2$, then $1 < q \leq 2$, and we have the embedding $L^p(B_n) \hookrightarrow L^q(B_n)$. According to (2.13) – (2.15), we get

$$\begin{aligned}\int_0^T \int_{B_n} \mid f_1(u) \mid^q \quad &\leq \quad C\int_0^T \int_{B_n} (\mid u_n \mid^{p-1} + \mid u_n \mid)^q \mathrm{d}x\mathrm{d}t \\ &\leq \quad C\int_0^T \int_{B_n} \mid u_n \mid^{(p-1)q} \mathrm{d}x\mathrm{d}t + C\int_0^T \int_{B_n} \mid u_n \mid^q \mathrm{d}x\mathrm{d}t \\ &\leq \quad C\int_0^T \int_{B_n} \mid u_n \mid^p + C\int_0^T \int_{B_n} \mid u_n \mid^p \mathrm{d}x\mathrm{d}t \\ &\leq \quad C. \end{aligned} \tag{2.16}$$

$$\begin{aligned}\int_0^T \int_{B_n} \mid f_2(u) \mid^q \quad &\leq \quad C\int_0^T \int_{B_n} \mid a(x) \mid^q (\mid u_n \mid^{p-1} + 1)^q \mathrm{d}x\mathrm{d}t \\ &\leq \quad C \mid a(x) \mid_\infty^{q-1} \int_0^T \int_{B_n} a(x)(\mid u_n \mid^{(p-1)q} + 1)\mathrm{d}x\mathrm{d}t \\ &\leq \quad C \mid a(x) \mid_\infty^{q-1} (C \mid a(x) \mid_1 + \int_0^T \int_{B_n} a(x) \mid u_n \mid^p \mathrm{d}x\mathrm{d}t) \\ &\leq \quad C. \end{aligned} \tag{2.17}$$

where C is independent on n.

Thanks to (2.16)−(2.17), $f_1(u_n)$ is bounded in $L^p(0,T;L^q(B_n))$, and $af_2(u_n)$ is bounded in $L^p(0,T;L^q(B_n))$.

For $\forall\ v \in L^2(0,T;H_0^1(B_n))$,

$$\begin{aligned}\int_0^T\int_{B_n} -\Delta u_n v &= \int_0^T\int_{B_n} \nabla u_n \nabla v \\ &\le (\int_0^T \parallel \nabla u_n \parallel_{B_n}^2)^{\frac{1}{2}}(\int_0^T \parallel \nabla v \parallel_{B_n}^2)^{\frac{1}{2}} \\ &\le (\int_0^T \parallel \nabla u_n \parallel^2)^{\frac{1}{2}}(\int_0^T \parallel \nabla v \parallel_{B_n}^2)^{\frac{1}{2}} \\ &\le C \parallel \nabla v \parallel_{H_0^1(B_n)}. \end{aligned} \tag{2.18}$$

where C is independent on n. We can obtain $-\Delta u_n$ is bounded in $L^2(0,T;H^{-1}(B_n))$.

Since $g(x) \in L^2(R^N)$, so

$$g(x) \in L^2(0,T;R^N). \tag{2.19}$$

Hence there exists $s > 0$, such that $L^2(0,T;H^{-1}(B_n))$, $L^2(0,T;H_0^1(B_n))$, $L^q(0,T;L^q(B_n))$,$L^2(0,T;L^2(B_n))$ are continuous embedding to $L^q(0,T;H^{-s}(B_n))$.

According to (2.5), (2.16) − (2.19), we obtain

$$u_{nt} - \Delta u_{nt} \in L^q(0,T;H^{-s}(B_n)). \tag{2.20}$$

Hence u_n has a subsequence (we also denote u_n) weak* convergence to u in $L^2(0,T;H^{-1}(B_n))$ and $L^2(0,T;L^2(B_n))$, $u_{nt} - \Delta u_{nt}$ has a subsequence (we also denote $u_{nt} - \Delta u_{nt}$) weak* convergence to $u_t - \Delta u_t$. Let $u_n = 0$ outside of B_n, we can extend u_n to R^N.

As introduced in [3,11], $C_c^\infty(R^N)$ is dense in the dual space of $H^{-1}(B_n))$, $L^2(B_n)$, $L^q(B_n)$ and $H^{-s}(B_n)$, so we can choose $\forall\ \phi \in L^2(0,T;C_c^\infty(R^N)) \cap L^q(0,T;C_c^\infty(R^N))$ as a test function such that

$$\langle \Delta u_n, \phi\rangle \to \langle \Delta u, \phi\rangle \tag{2.21}$$

$$\langle u_{nt} - \Delta u_{nt}, \phi\rangle \to \langle u_t - \Delta u_t, \phi\rangle \tag{2.22}$$

Since $\forall\ \phi \in C_c^\infty(R^N)$, there exists bounded domain $\Omega \subset R^N$ such that $\phi = 0,\ x \notin \Omega$. Hence u_n is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$, and $u_{nt} - \Delta u_{nt} \in L^q(0,T;H^{-s}(\Omega))$. Since $H_0^1(\Omega) \subset\subset L^2(\Omega) \subset H^{-s}(\Omega)$, according to lemma 2.1, there is a subsequence u_n (we also denote u_n) that converges strongly to u in $L^2(0,T;L^2(\Omega))$.

Using the continuity of f_1 and f_2, we have

$$\langle f_1(u_n), \phi\rangle \to \langle f_1(u), \phi\rangle \tag{2.23}$$

$$\langle a(x)f_2(u_n), \phi\rangle \to \langle a(x)f_2(u), \phi\rangle \tag{2.24}$$

According to (2.21) − (2.24), and let $n \to \infty$, we get : $\forall\ \phi \in L^2(0,T;C_c^\infty(R^N)) \cap L^q(0,T;C_c^\infty(R^N))$,

$$\langle u_t - \Delta u_t - \Delta u + f_1(u) + a(x)f_2(u), \phi\rangle = \langle g(x), \phi\rangle \tag{2.25}$$

Ref.

[3] V. K. Kalantarov, *On the attractors for some non-linear problems of mathematical physics*, Zap. Nauch. Sem. LOMI **152** (1986):50-54.
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Hence u is the weak solution of (2.1) − (2.3) and satisfy

$$u \in \mathcal{C}^1([0,T]; H^1(R^N)) \cap L^p(0,T; L^p(R^N)).$$

Step 3 Uniqueness and continuous dependence.

Let u_0, v_0 be in $H^1(R^N)$, and setting $w(t) = u(t) - v(t)$, we see that $w(t)$ satisfies

$$w_t - \Delta w_t - \Delta w + f_1(u) - f_1(v) + a(x)(f_2(u) - f_2(v)) = 0, \quad x \in R^N. \tag{2.26}$$

Notes

Taking the inner product with w of (2.26), using (F_1), (F_2) and (A), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\| \nabla w \|^2 + \|w\|^2) + \| \nabla w \|^2 &\leq | \int (f_1(u) - f_1(v)) w \mathrm{d}x | + | \int a(x)(f_2(u) - f_2(v)) w \mathrm{d}x | \\ &\leq C(1 + \| a \|_\infty) \| w \|^2 \end{aligned} \tag{2.27}$$

By the Gronwall Lemma, we get

$$\| \nabla w(t) \|^2 + \|w(t)\|^2 \leq e^{Ct}(\| \nabla w(0) \|^2 + \|w(0)\|^2). \tag{2.28}$$

This is uniqueness and is continuous dependence on initial conditions.

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